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Subject

ICE FORMATION IN THE ARCTIC DURING SUMMER: FALSE-BOTTOMS

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CONTENT

1 Introduction

There exists fresh water in the Arctic during summer. First, melt water collects in surface melt pond (melting under the sun) which is the most important reservoir. Second, melt water can percolate into the ice matrix to form an under-ice melt pond. At the interface between this fresh water and the underlying salt water, double-diffusive convection of heat and salt occurs, leading to the formation of underwater ice called "false-bottoms" (see, e.g, [2, 5, 6]). Note that salt water has the double properties:

i) it does not freeze even for temperature $< 0^0$C,

ii) it dissolves ice when it is in contact with ice.

Thus false-bottoms is a layer of ice, which relatively protects the fresh water from being mixed into the underlying salt water. Such false-bottoms is the only significant source of ice formation in the Arctic during summer.

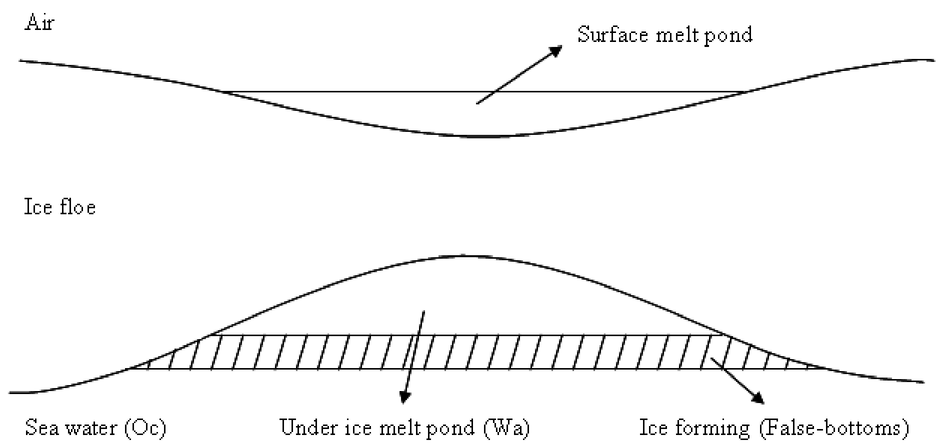

Figure 1: Ice formation in the Arctic during summer.

The main difference between the false-bottoms case and the normal sea ice case is the presence of the growth at the upper interface of the first one.

1.1 One-dimensional model

We consider a one-dimensional model describing the simultaneous growth and ablation of the ice of false-bottoms. Here we have three environments: the ocean (Oc), the ice of false-bottoms (Fb) and the fresh water (Wa). Denote by $T(x,t)$, $S(x,t)$ the temperature and the salinity, and denote by $h_0(t), h_u(t)$ the free boundaries at the interfaces ice-ocean (Fb-Oc) and ice-water (Fb-Wa), respectively.

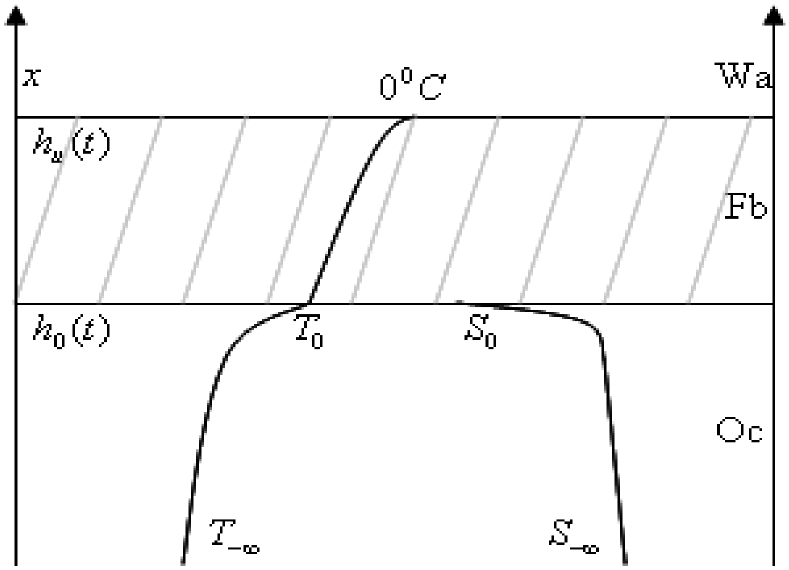

Figure 2: The one-dimensional model.

At the interface Fb-Oc, we apply the first principle of thermodynamics, i.e.,

$$\Delta U = \Delta \Phi$$

(variation of energy = variation of heat flux through the interface).
The net amount of heat transferred through the interface Fb-Oc in a section s is equal to

$$\Delta U = -dh_0 \rho_I L_f s,$$

where ρ_I is the density of the ice and L_f is the latent heat of fusion. On the other hand, the difference of the heat fluxes through a section s in the ice and the ocean during a time dt is

$$\Delta \Phi = \left(-\lambda_I \left.\frac{\partial T}{\partial x}\right]_{Fb,h_0} + \lambda_O \left.\frac{\partial T}{\partial x}\right]_{Oc,h_0} \right) s dt,$$

where λ_I, λ_O are thermal conductivities of the ice and the ocean. Finally, the law of conservation of energy, i.e. $\Delta U = \Delta \Phi$, leads to the Stefan condition for the heat balance at the interface

$$h_0'(t) = \widetilde{\lambda}_I \left.\frac{\partial T}{\partial x}\right]_{Fb,h_0} - \widetilde{\lambda}_O \left.\frac{\partial T}{\partial x}\right]_{Oc,h_0}, \tag{1.1}$$

where

$$\widetilde{\lambda}_I = \frac{\lambda_I}{\rho_I L_f},\ \widetilde{\lambda}_O = \frac{\lambda_O}{\rho_I L_f}.$$

For simplicity, we can neglect the salt of the ice of false-bottoms. The water near the interface Fb-Oc is a mixture of melt water, which melts from the ice of false-bottoms, and sea water. This water freshens at the rate

$$S_0(t) h_0'(t),$$

while salt diffuses into this water at the rate

$$-D \left.\frac{\partial S}{\partial x}\right]_{Oc,h_0},$$

where $S_0(t) = S(h_0(t), t)$ is the salinity of the ocean at the interface and D is the molecular diffusivity of salt in sea water. The balance of salt at this interface leads to the conservation condition

$$S_0(t)h_0'(t) = -D \left.\frac{\partial S}{\partial x}\right]_{Oc,h_0}. \tag{1.2}$$

The establishment of these equations bases on Martin and Kauffman [5] and Notz et al. [6].

At the interface Fb-Wa, we use the simplified scheme in [4], in which the temperature of the water in the under-ice melt pond is kept at 0^0C. In particular, it leads to the boundary condition at the interface

$$T(h_u(t), t) = 0. \tag{1.3}$$

Furthermore, due to the neglect of heat fluxes from the fresh water above the false-bottoms, a thermodynamic condition similar to (1.1) reduces to the Stefan condition at the upper surface

$$h_u'(t) = \widetilde{\lambda}_I \left.\frac{\partial T}{\partial x}\right]_{Fb,h_u}. \tag{1.4}$$

Equations (1.1),(1.2),(1.4) form a closed system in connection with the diffusion equations for heat and salt in the ice and in the ocean

$$\left.\frac{\partial T}{\partial t}\right]_{Fb} = D_I \left.\frac{\partial^2 T}{\partial x^2}\right]_{Fb}, \ h_u(t) > x > h_0(t), \tag{1.5}$$

$$\left.\frac{\partial T}{\partial t}\right]_{Oc} = D_O \left.\frac{\partial^2 T}{\partial x^2}\right]_{Oc}, \ h_0(t) > x > -\infty, \tag{1.6}$$

and

$$\left.\frac{\partial S}{\partial t}\right]_{Oc} = D \left.\frac{\partial^2 S}{\partial x^2}\right]_{Oc}, \ h_0(t) > x > -\infty, \tag{1.7}$$

where D_I and D_O are the thermal diffusivity. Here all constants are positive.

The problem is of finding (h_0, h_u, T, S) from the system (1.1)-(1.7), where $T = T(x,t)$ with $h_u(t) > x > -\infty$ and $S = S(x,t)$ with $h_0(t) > x > -\infty$.

1.2 Numerical solution

Now we consider a special solution (h_0, h_u, T, S) of the system (1.1)-(1.7), which is useful for physical application. Assume that we have known the initial conditions $h_0(0) = 0$, $h_u(0) = a_0 > 0$, and the far-field information $T(-\infty, t) = T_{-\infty}(t)$ and

$S(-\infty,t) = S_{-\infty}(t)$. Moreover, we shall use the freezing-point relationship between the interface temperature $T_0(t)$ and the salinity $S_0(t)$

$$T_0 \approx -mS_0, \tag{1.8}$$

where $m = 0.054^0\text{C psu}^{-1}$. This is an approximation for the liquidus temperature [6].

The brief idea is as follows. We shall find special solutions of diffusion equations (1.5)-(1.7) under the form of functions in term of $\frac{x}{2\sqrt{t}}$ only. Under this form, the solutions T and S are determined completely in term of the free boundaries $h_0(t), h_0(t)$ and the boundary conditions $(T_{-\infty}, T_0, S_{-\infty}, S_0)$. Moreover, S_0 is determined from T_0 by the relationship (1.8). Therefore, the original problem of finding (h_0, h_u, T, S) from (1.5)-(1.8) reduces to a problem of finding $(h_0(t), h_u(t), T_0(t))$ from a system of nonlinear differential equations.

To go into the details, by a change of variable we first choose $t_0 > 0$ small, and then consider the problem for $t > t_0$ and $h_0(t_0) = 0$, $h_u(t_0) = a_0$. We shall now find a solution of the heat equation (1.6) under the form

$$T(x,t) = \widetilde{T}(v), \ v = \frac{x}{2\sqrt{D_O t}}.$$

By direct computation, one has

$$\frac{\partial T}{\partial t} = -\frac{x}{4\sqrt{D_O}t^{3/2}}\widetilde{T}'(v), \ \frac{\partial T}{\partial x} = \frac{1}{2\sqrt{D_O t}}\widetilde{T}'(v), \ \frac{\partial^2 T}{\partial x^2} = \frac{1}{4D_O t}\widetilde{T}''(v).$$

Therefore (1.6) reduces to

$$2v\widetilde{T}'(v) + \widetilde{T}''(v) = 0, \text{ that is } \left(\widetilde{T}'(v)e^{v^2}\right)' = 0.$$

Thus $\widetilde{T}'(v) = \Lambda e^{-v^2}$, where Λ is independent on v, and hence

$$\frac{\partial T}{\partial x}(x,t) = \frac{1}{2\sqrt{D_O t}}\widetilde{T}'(v) = \frac{\Lambda}{2\sqrt{D_O t}}e^{-v^2}.$$

To find Λ in term of $(T_{-\infty}, T_0, h_0)$, we use the boundary conditions $T(h_0(t),t) = T_0$ and $T(-\infty,t) = T_{-\infty}$ as follows

$$T_0 - T_{-\infty} = \int_{-\infty}^{h_0(t)} \frac{\partial T}{\partial x} dx = \int_{-\infty}^{h_0(t)} \frac{a}{2\sqrt{D_O t}} e^{-v^2} dx = \Lambda \int_{-\infty}^{\frac{h_0(t)}{2\sqrt{D_O t}}} e^{-v^2} dv.$$

Introduce the function

$$F(x) = \int_{-\infty}^{x} e^{-r^2} dr = \frac{\pi}{2} + \frac{\pi}{2}\text{erf}(x).$$

Then

$$\left.\frac{\partial T}{\partial x}\right]_{Oc,h_0} = \frac{a}{2\sqrt{D_O t}} e^{-v^2(h_0(t),t)} = \frac{T_0 - T_{-\infty}}{2\sqrt{D_O t}} \frac{\exp\left(-\frac{h_0^2}{4D_O t}\right)}{F\left(\frac{h_0}{2\sqrt{D_O t}}\right)}. \tag{1.9}$$

Similarly, we find solutions of equations (1.7) and (1.5) under the form $S(x,t) = \widetilde{S}(v)$ and $T(x,t) = \widetilde{T}(v)$, $v = \frac{x}{2\sqrt{t}}$, respectively. Equation (1.7), with the boundary conditions $S(h_0(t),t) = S_0$ and $S(-\infty,t) = S_{-\infty}$, gives

$$\left.\frac{\partial S}{\partial x}\right]_{Oc,h_0} = \frac{S_0 - S_{-\infty}}{2\sqrt{Dt}} \frac{\exp\left(-\frac{h_0^2}{4Dt}\right)}{F\left(\frac{h_0}{2\sqrt{Dt}}\right)}, \tag{1.10}$$

and equation (1.5), with the boundary conditions $T(h_0(t),t) = T_0(t)$ and $T(h_u(t),t) = 0$, gives

$$\begin{aligned} \left.\frac{\partial T}{\partial z}\right]_{Fb,h_0} &= \frac{T_0}{2\sqrt{D_I t}} \frac{\exp\left(-\frac{h_0^2}{4D_I t}\right)}{F\left(\frac{h_0}{2\sqrt{D_I t}}\right) - F\left(\frac{h_u}{2\sqrt{D_I t}}\right)}, \\ \left.\frac{\partial T}{\partial z}\right]_{Fb,h_u} &= \frac{T_0}{2\sqrt{D_I t}} \frac{\exp\left(-\frac{h_u^2}{4D_I t}\right)}{F\left(\frac{h_0}{2\sqrt{D_I t}}\right) - F\left(\frac{h_u}{2\sqrt{D_I t}}\right)}. \end{aligned} \tag{1.11}$$

Substituting (1.9), (1.10), (1.11) into (1.1), (1.2), (1.4), and using (1.8) to replace S_0 by $-T_0/m$, we obtain

$$\begin{cases} h_0'(t) = \widetilde{\lambda}_I \dfrac{T_0}{2\sqrt{D_I t}} \dfrac{\exp\left(-\frac{h_0^2}{4D_I t}\right)}{F\left(\frac{h_0}{2\sqrt{D_I t}}\right) - F\left(\frac{h_u}{2\sqrt{D_I t}}\right)} - \widetilde{\lambda}_O \dfrac{T_0 - T_{-\infty}}{2\sqrt{D_O t}} \dfrac{\exp\left(-\frac{h_0^2}{4D_O t}\right)}{F\left(\frac{h_0}{2\sqrt{D_O t}}\right)}, \\ h_0'(t) = -\left(1 + \dfrac{mS_{-\infty}}{T_0}\right) \dfrac{\sqrt{D}}{2\sqrt{t}} \dfrac{\exp\left(-\frac{h_0^2}{4Dt}\right)}{F\left(\frac{h_0}{2\sqrt{Dt}}\right)}, \\ h_u'(t) = \widetilde{\lambda}_I \dfrac{T_0}{2\sqrt{D_I t}} \dfrac{\exp\left(-\frac{h_u^2}{4D_I t}\right)}{F\left(\frac{h_0}{2\sqrt{D_I t}}\right) - F\left(\frac{h_u}{2\sqrt{D_I t}}\right)}. \end{cases} \tag{1.12}$$

Solving this nonlinear system, we get a numerical solution of $(h_0(t), h_u(t), T_0(t))$. To go into the details, we eliminate $h_0'(t)$ in the two first equations of (1.12) to get

$$A(t, h_0, h_u)T_0^2 + B(t, h_0)T_0 + C(t, h_0) = 0$$

where

$$\begin{aligned} A(t, y_1, y_2) &= \frac{\widetilde{\lambda}_I}{\sqrt{D_I}} \frac{\exp\left(-\frac{y_1^2}{4D_I t}\right)}{F\left(\frac{y_1}{2\sqrt{D_I t}}\right) - F\left(\frac{y_2}{2\sqrt{D_I t}}\right)} - \frac{\widetilde{\lambda}_O}{\sqrt{D_O}} \frac{\exp\left(-\frac{y_1^2}{4D_O t}\right)}{F\left(\frac{y_1}{2\sqrt{D_O t}}\right)}, \\ B(t, y_1) &= \frac{\widetilde{\lambda}_O T_{-\infty}}{\sqrt{D_O}} \frac{\exp\left(-\frac{y_1^2}{4D_O t}\right)}{F\left(\frac{h_1}{2\sqrt{D_O t}}\right)} + \sqrt{D} \frac{\exp\left(-\frac{y_1^2}{4Dt}\right)}{F\left(\frac{y_1}{2\sqrt{Dt}}\right)}, \\ C(t, y_1) &= -\sqrt{D} m S_{-\infty} \frac{\exp\left(-\frac{y_1^2}{4Dt}\right)}{F\left(\frac{y_1}{2\sqrt{Dt}}\right)}. \end{aligned}$$

Note that $A(t, h_0, h_u) < 0$ (since $h_0 < h_u$) and $C(t, h_0) > 0$. Thus, T_0 is the unique negative solution of the latter equation and is given by

$$T_0(t) = \frac{-B(t,h_0) + \sqrt{B^2(t,h_0) - 4A(t,h_0,h_u)C(t,h_0)}}{2A}. \tag{1.13}$$

Replacing (1.13) into the first and the third equations of (1.12), we get

$$\begin{cases} h_0'(t) = L_1(t,h_0,h_u), & t > 0, \\ h_u'(t) = L_2(t,h_0,h_u), & t > 0, \end{cases} \tag{1.14}$$

where

$$\begin{aligned} L_1(t,y_1,y_2) &= \frac{-B(t,y_1) + \sqrt{B^2(t,y_1) - 4A(t,y_1,y_2)C(t,y_1)}}{4\sqrt{t}} + \\ &+ \widetilde{\lambda}_O \frac{T_{-\infty}}{2\sqrt{D_O t}} \frac{\exp\left(-\frac{y_1^2}{4D_O t}\right)}{F\left(\frac{y_1}{2\sqrt{D_O t}}\right)}, \\ L_2(t,y_1,y_2) &= \widetilde{\lambda}_I \frac{\left(-B(t,y_1) + \sqrt{B^2(t,y_1) - 4A(t,y_1,y_2)C(t,y_1)}\right)}{4A(t,y_1,y_2)\sqrt{D_I t}} \times \\ &\times \frac{\exp\left(-\frac{y_2^2}{4D_G t}\right)}{F\left(\frac{y_1}{2\sqrt{D_I t}}\right) - F\left(\frac{y_2}{2\sqrt{D_I t}}\right)}. \end{aligned}$$

It is clear that $\Omega = \{(t, y_1, y_2) \in R^3 |\ t > 0, y_2 > y_1\}$ is an open subset in R^3 and $L_1, L_2 \in C^\infty(\Omega)$, and hence in particular L_1, L_2 are locally Lipschitz in Ω. Therefore, from $(t_0, 0, a_0) \in \Omega$ then the problem (1.14) has a unique maximum solution (h_0, h_u) such that $h_0(t_0) = 0$, $h_u(t_0) = a_0 > 0$, and $h_u(t) > h_0(t)$. Of course $T_0(t)$ follows (1.13). We have thus proved the following result.

Theorem 1.1. *Let $T_{-\infty}(t)$, $S_{-\infty}(t)$ be continuous functions and let $t_0 > 0$, $a_0 > 0$ be constants. Then the system (1.12) has a unique solution $(h_0(t), h_u(t), T_0(t))$ in $t \in (t_0, T^*)$ for some $T^* > t_0$ such that*

$$h_0(t_0) = 0,\ \ h_u(t_0) = a_0,\ \ h_u(t) > h_0(t),\ \ \text{and}\ T_0(t) < 0,\ \ \forall t \in (t_0, T^*).$$

Moreover, this solution can be extended uniquely by prolongation whenever the condition $h_u(T^*) > h_0(T^*)$ still holds.

Remark 1.1. *The system (1.12) can be solved by a numerical routine. Although the program does not run with the initial condition $h_u(0) = h_0(0) = 0$, we can choose $t_0 > 0$ and $a_0 > 0$ small to simulate the phenomenon from initial time. This situation is similar to a physical experiment given by Notz et al. [6]. In which, the authors first put fresh water at 0^0C on top of salt water in order to simulate the evolution of a false-bottoms, but this model does not take salt transport through the false-bottom into account. To adjust, they started with a 5-cm layer of ice, which formed on day 15 in the experiment, and let the model run until day 35 of the experiment.*

Here are some figures of a such solution.

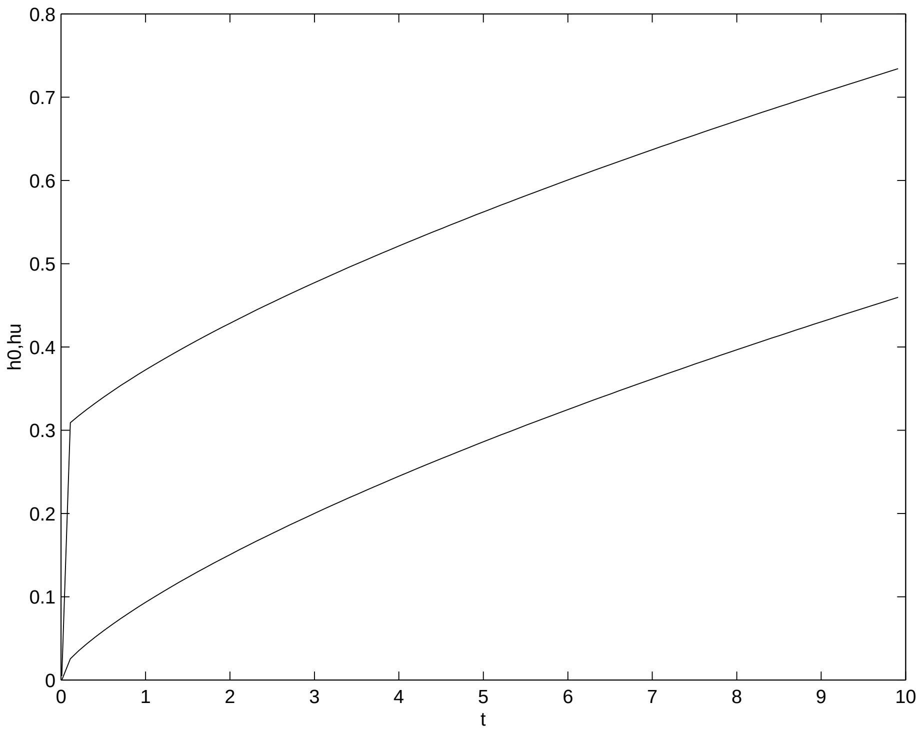

Figure 3: The free boundaries $h_0(t)$ and $h_u(t)$.

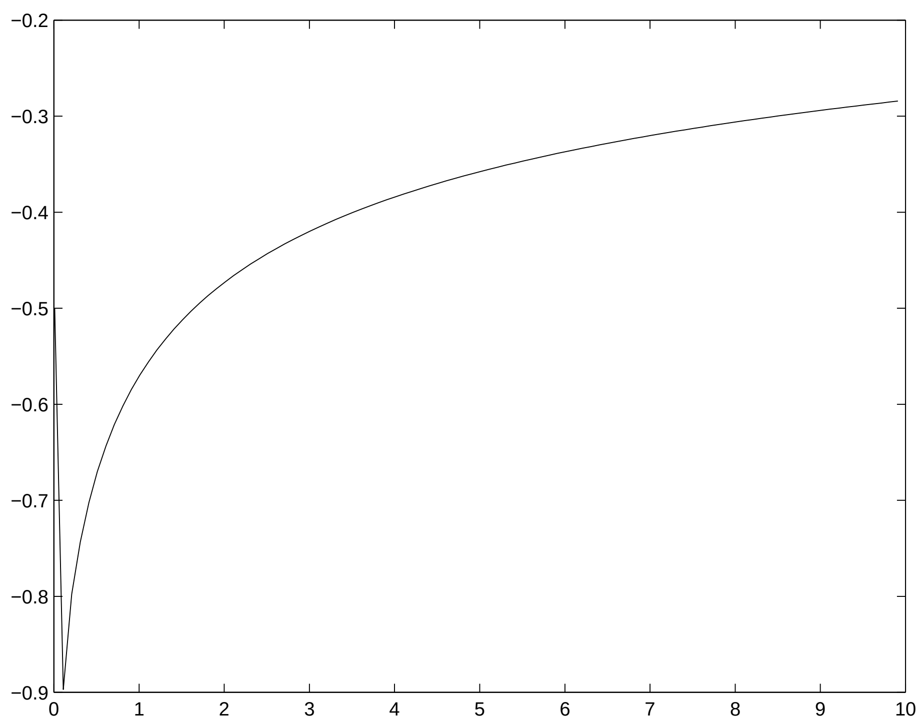

Figure 4: The freezing temperature $T_0(t)$.

Remark 1.2. *In Fig.3 and Fig.4, the functions h_0, h_u and T_0 change rapidly at the initial time and then increase gradually. This result is compatible with the physical* *background of the problem. The increase of both $h_u(t)$ and $h_0(t)$ corresponds to the simultaneous growth and ablation of the ice of false-bottoms. Here, we see the effect of two properties of salt water: $T_0(t) < 0$ implies that the temperature near the interface h_0 is negative but the sea water does not freeze, and the growth of $h_0(t)$ implies that sea water dissolves the ice when it is in contact with ice.*

Remark 1.3. *The system (1.12) gives a numerical solution of the system (1.1)-(1.7), in which the function is dependent only on $x/\sqrt{t}$. It is reasonable for a long-term observation. As known [1], since there are no external length-scales or time-scales in the problem, the system (1.1)-(1.7) admits a similarity solution depending only on $x/\sqrt{t}$ in unbounded domain. Furthermore, A. Friedmann ([3], Chap.8) said that the solutions of Stefan problems asymptotically depends only on $x/\sqrt{t}$ when $t \to +\infty$.*

Remark 1.4. *In physical applications, ones even assume more that $h_0(t) = \lambda_0\sqrt{t}$ and $h_u(t) = \lambda_u\sqrt{t}$, for constants λ_0, λ_u, to keep the analysis simple [6]. Although these simplicities make the system (1.1)-(1.7) having no solution in really mathematical sense, they still produce a "solution" useful for the long term observation. Infact, the time is longer, our result given in Fig.3 fits better with the laboratory experiment given by Martin and Kauffman[5] and Notz et al. [6].*

2 Existence and Uniqueness

In this section, we consider the problem (1.1)-(1.7) mathematically. We want to give the existence and uniqueness of the solution (h_0, h_u, T, S) under suitable data. For this purpose, it is reasonable to require the initial conditions $h_0(0)$, $h_u(0)$, $T(x,0) = T^0(x)$ with $h_u(0) > x > -\infty$, and $S(x,0) = S^0(x)$ with $h_0(0) > x > -\infty$. Moreover, assume that the freezing temperature $T(h_0(t), t) = T_0(t)$ with $0 \le t \le \sigma$ has been known .

For rigour, we need some assumption on the data as well as a definition of the solution.

Hypothesis 2.1. *Let $\sigma > 0$. Assume that*

(H1) $h_0(0) < h_u(0)$.

(H2) $T^0(x)$ is continuous at $x = h_0(0)$, $x = h_u(0)-$, and satisfies the compatible conditions $T^0(h_0(0)) = T_0(0)$, $T^0(h_u(0)) = 0$; $T^0_x(x)$ is continuous and bounded in $(-\infty, h_0(0))$ and $(h_0(0), h_u(0))$.

(H3) $S^0(x)$ is continuous and bounded in $(-\infty, h_0(0))$.

(H4) $T_0(t)$ is continuously differentiable in $0 \le t \le \sigma$.

Definition 2.1. *We say (h_0, h_u, T, S) is a solution in $0 \le t \le \sigma$ if*

(C1) $h_0(t)$, $h_u(t)$ is continuously differentiable and $h_u(t) > h_0(t)$ in $0 \le t < \sigma$.

(C2) T_t, T_{xx} is continuous in $0 < t \le \sigma$, $h_u(t) > x > h_0(t)$ and $h_0(t) > x > -\infty$; $T(x,.)$ is continuous at $t = 0$; $T(.,t)$ is continuous at $x = h_0(t)$ and $x = h_u(t)-$; $T_x(.,t)$ is continuous at $x = h_0(t)-$, $x = h_0(t)+$ and $x = h_u(t)-$.

(C3) S_t, S_{xx} is continuous in $0 < t \le \sigma$, $h_0(t) > x > -\infty$; $S(x,.)$ is continuous at $t = 0$; $S(.,t)$ is continuous at $x = h_0(t)$.

(C4) The system (1.1)-(1.7) is satisfied in $0 < t < \sigma$ with the initial conditions $(h_0(0), h_u(0), T^0(x), S^0(x))$ and the freezing temperature $T(h_0(t), t) = T_0(t)$.

The main result in this section is as follows.

Theorem 2.1. *Assume that $(h_0(0), h_u(0), T^0(x), S^0(x), T_0(t))$ satisfy Hypothesis (H1)-(H4) for some $\sigma > 0$. Then the system (1.1)-(1.7) has a unique solution (h_0, h_u, T, S) in $0 \le t \le \sigma^*$ for some $\sigma^* \in (0, \sigma]$. Moreover, this solution can be extended uniquely whenever the condition $h_u(\sigma^*) > h_0(\sigma^*)$ still holds.*

Our problem is a two-phase Stefan problem with two free boundaries. We shall follow the approach of A.Friedman ([3], Chap.8), which deals with a classical one-phase Stefan problem with one free boundary. We first reduce the problem to solving a system of integral equations with respect to $S_0(t) = S(h_0(t)-, t)$, $v_1(t) = T_x(h_0(t)-, t)$, $v_2(t) = T_x(h_0(t)+, t)$ and $v_3(t) = T_x(h_u(t)-, t)$, and then solve this system by contraction principle.

2.1 Preliminaries

Let $a(t), b(t)$ be continuously differentiable functions and $b(t) > a(t)$, $t \ge 0$. Let $\kappa > 0$ be a constant and let $u(x,t)$ be a solution of the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \; b(t) > x > a(t), \tag{2.1}$$

with the initial condition $u(x,0)$ bounded in $(a(0),b(0))$.

Introduce the Green's function of equation (2.1),

$$G(x,t;\xi,\tau)=\frac{H(t-\tau)}{2\sqrt{\pi\kappa(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right),$$

where H is Heaviside function,

$$H(t)=\begin{cases}1, & \text{if } t>0,\\ 0, & \text{if } t<0.\end{cases}$$

We have the following Lemma to represent the solution of (2.1) from its initial condition and boundary conditions.

Lemma 2.1. *For $t>0$ and $a(t)<x<b(t)$ we have*

$$\begin{aligned}u(x,t) &= \int_0^t G(x,t;b(\tau),\tau)\left[\kappa u_\xi(b(\tau)-,\tau)+u(b(\tau),\tau)b'(\tau)\right]d\tau\\ &\quad-\int_0^t G(x,t;a(\tau),\tau)\left[\kappa u_\xi(a(\tau)+,\tau)+u(a(\tau),\tau)a'(\tau)\right]d\tau\\ &\quad-\kappa\int_0^t G_\xi(x,t;b(\tau),\tau)u(b(\tau),\tau)d\tau+\kappa\int_0^t G_\xi(x,t;a(\tau),\tau)u(a(\tau),\tau)d\tau\\ &\quad+\int_{a(0)}^{b(0)} G(x,t;\xi,0)u(\xi,0)d\xi.\end{aligned}$$

Proof. Note that

$$G_\tau+\kappa G_{\xi\xi}=0 \text{ for all } \tau<t, \quad \text{and } G(x,t;\xi,t-)=\delta(x-\xi), \tag{2.2}$$

where $\delta=H'$ is Dirac delta function.

Integrating the Green's identity, here $u=u(\xi,\tau)$,

$$\kappa\frac{\partial}{\partial\xi}\left(G\frac{\partial u}{\partial\xi}-u\frac{\partial G}{\partial\xi}\right)-\frac{\partial}{\partial\tau}(uG)=0$$

over the domain $a(\tau)<\xi<b(\tau), 0<\tau<t$, we will obtain the desired result because

$$\begin{aligned}\int_0^t\int_{a(\tau)}^{b(\tau)}\frac{\partial}{\partial\xi}\left(G\frac{\partial u}{\partial\xi}-u\frac{\partial K}{\partial\xi}\right)d\xi d\tau &= \int_0^t\left[G\frac{\partial u}{\partial\xi}-u\frac{\partial G}{\partial\xi}\right]_{\xi=a(\tau)}^{\xi=b(\tau)}d\tau\\ =\int_0^t G(x,t;b(\tau),\tau)u_\xi(b(\tau)-,\tau)d\tau &- \int_0^t G(x,t;a(\tau),\tau)u_\xi(a(\tau)+,\tau)d\tau\\ -\int_0^t G_\xi(x,t;b(\tau),\tau)u(b(\tau),\tau)d\tau &+ \int_0^t G_\xi(x,t;a(\tau),\tau)u(a(\tau),\tau)d\tau,\end{aligned}$$

and

$$\int_0^t \int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau}(uG)\, d\xi d\tau$$

$$= \int_0^t \left\{ \frac{\partial}{\partial \tau} \left(\int_{a(\tau)}^{b(\tau)} uG d\xi \right) - [uG]_{\xi=b(\tau)} b'(\tau) + [uG]_{\xi=a(\tau)} a'(\tau) \right\} d\tau$$

$$= \left[\int_{a(\tau)}^{b(\tau)} uG d\xi \right]_{\tau=0}^{\tau=t-} - \int_0^t [uG]_{\xi=b(\tau)} b'(\tau) d\tau + \int_0^t [uG]_{\xi=a(\tau)} a'(\tau) d\tau$$

$$= u(x,t) - \int_{a(0)}^{b(0)} G(x,t;\xi,0) u(\xi,0) d\xi - \int_0^t G(x,t;b(\tau),\tau) u(b(\tau),\tau) b'(\tau) d\tau$$

$$+ \int_0^t G(x,t;a(\tau),\tau) u(a(\tau),\tau) a'(\tau) d\tau.$$

Here we have made use the following equality, with $v = uG$,

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi = \frac{\partial}{\partial \tau} \left(\int_{a(\tau)}^{b(\tau)} v(\xi,\tau) d\xi \right) - v(b(\tau),\tau) b'(\tau) + v(a(\tau),\tau) a'(\tau).$$

In fact, in the case $a(t) \equiv 0$ we have

$$\int_0^{b(\tau)} \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi = \int_0^{\infty} H(b(\tau)-\xi) \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi$$

$$= \int_0^{\infty} \frac{\partial}{\partial \tau}[H(b(\tau)-\xi) v(\xi,\tau)]\, d\xi - \int_0^{\infty} \frac{\partial}{\partial \tau}[H(b(\tau)-\xi)]\, v(\xi,\tau) d\xi$$

$$= \frac{\partial}{\partial \tau} \left(\int_0^{\infty} H(b(\tau)-\xi) v(\xi,\tau) d\xi \right) - \int_0^{\infty} \delta(b(\tau)-\xi) b'(\tau) v(\xi,\tau) d\xi$$

$$= \frac{\partial}{\partial \tau} \left(\int_0^{b(\tau)} v(\xi,\tau) d\xi \right) - v(b(\tau),\tau) b'(\tau).$$

If $a(t)$ is not constant, we can write

$$\int_{a(\tau)}^{b(\tau)} \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi = \int_0^{b(\tau)} \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi - \int_0^{a(\tau)} \frac{\partial}{\partial \tau}(v(\xi,\tau))\, d\xi,$$

and the desired equality follows. This completes the proof. □

Remark 2.1. *The result in Lemma 2.1 still holds for $b(t) \equiv +\infty$ or $a(t) \equiv -\infty$. For example, if $a(t) \equiv -\infty$ then the fomular in Lemma 2.1 reduces to*

$$\begin{aligned} u(x,t) &= \int_0^t G(x,t;b(\tau),\tau)\left[\kappa u_\xi(b(\tau)-,\tau) + u(b(\tau),\tau)b'(\tau)\right] d\tau \\ &\quad -\kappa \int_0^t G_\xi(x,t;b(\tau),\tau)u(b(\tau),\tau)d\tau + \int_{-\infty}^{b(0)} G(x,t;\xi,0)u(\xi,0)d\xi. \end{aligned}$$

We shall need the following Lemma (see [3], page 217, Lemma 1).

Lemma 2.2. *Let $p(t)$ be continuous and let $s(t) > 0$ satisfy the Lipschitz condition, $0 \le t \le \sigma$. Then, for $0 < t \le \sigma$,*

$$\lim_{x \to s(t)^-} \kappa \int_0^t p(t)G_x(x,t;s(\tau),\tau)d\tau = \frac{1}{2}p(t) + \kappa \int_0^t p(t)G_x(s(t),t;s(\tau),\tau)d\tau.$$

Remark 2.2. *This lemma gives the jump relation at the boundary $s(t)$. In applications later, sometimes we need to note that $G_\xi = -G_x$. Moreover, for the right limit we have*

$$\lim_{x \to s(t)^+} \kappa \int_0^t p(t)G_x(x,t;s(\tau),\tau)d\tau = -\frac{1}{2}p(t) + \kappa \int_0^t p(t)G_x(s(t),t;s(\tau),\tau)d\tau.$$

Also, we need a simple version of the uniqueness for the system of linear Volterra integral equations of the second kind.

Lemma 2.3. *Let $n \in \mathbb{N}$, $\sigma > 0$, $p > 1$, $q > 1$, $1/p + 1/q = 1$. Assume that $\tau \mapsto W_j(t,\tau)$ is measurable in $(0,t)$ for all $t \in (0,\sigma]$ and*

$$\int_0^t |W_j(t,\tau)|^p \, d\tau \le const., \ \forall t \in (0,\sigma], j = \overline{1,n}.$$

Then the system

$$\Psi_i(t) = \sum_{j=1}^n \left(\int_0^t W_j(t,\tau)\Psi_j(\tau)d\tau \right), \forall t \in (0,\sigma], j = \overline{1,n},$$

has a unique solution $\{\Psi_j\}_{j=1}^n = 0$ in $L^q(0,\sigma)$.

Proof. Using Holder inequality one has

$$\begin{aligned}
|\Psi_i(t)| &\le \sum_{j=1}^{n} \left| \int_0^t W_j(t,\tau)\Psi_j(\tau)d\tau \right| \\
&\le \sum_{j=1}^{n} \left(\int_0^t |W_j(t,\tau)|^p \, d\tau \right)^{1/p} \left(\int_0^t |\Psi_j(\tau)|^q \, d\tau \right)^{1/q} \\
&\le const. \sum_{j=1}^{n} \left(\int_0^t |\Psi_j(\tau)|^q \, d\tau \right)^{1/q}.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} |\Psi_j(\tau)|^q \le const. \sum_{j=1}^{n} \left(\int_0^t |\Psi_j(\tau)|^q \, d\tau \right) = const. \int_0^t \left(\sum_{j=1}^{n} |\Psi_j(\tau)|^q \right) d\tau,$$

and it follows from Gronwall's Lemma that $\sum_{j=1}^{n} |\Psi_j(\tau)|^q = 0$. Thus $\{\Psi_j\}_{j=1}^n = 0$. □

Remark 2.3. *In later application, we have*

$$|W_j(t,\tau)| \le \frac{const.}{\sqrt{t-\tau}}, \quad j = \overline{1,n}.$$

In this case, we can choose any $p \in (1,2)$ *in order to apply Lemma 2.3.*

2.2 Reduction to integral equations

Denote by G_1, G_2, G_3 the function G in Lemma 2.1 corresponding to $\kappa = D, D_O, D_I$, respectively.

Applying Lemma 2.1 to equation (1.7) with $h_0(t) > x > -\infty$ and using condition $S_0(t)h_0'(t) + DS_x(h_0(t)-,t) = 0$, one has

$$S(x,t) = -D \int_0^t G_{1\xi}(x,t;h_0(\tau),\tau)S_0(\tau)d\tau + \int_{-\infty}^{h_0(0)} G_1(x,t;\xi,0)S(\xi,0)d\xi. \tag{2.3}$$

Thus $S(x,t)$ is determined completely by h_0 and $S_0(t)$. Taking $x \to h_0(t)-$ in (2.3) and using the jump relation in Lemma 2.2, we get

$$S_0(t) = -2D \int_0^t G_{1\xi}(h_0(t),t;h_0(\tau),\tau)S_0(\tau)d\tau + 2 \int_{-\infty}^{h_0(0)} G_1(h_0(t),t;\xi,0)S(\xi,0)d\xi. \tag{2.4}$$

Next, apply Lemma 2.1 to equation (1.6) , for $h_0(t) > x > -\infty$,

$$T(x,t) = \int_0^t G_2(x,t;h_0(\tau),\tau)\left[D_O v_1(\tau) + T_0(\tau)h_0'(\tau)\right]d\tau$$
$$- D_O \int_0^t G_{2\xi}(x,t;h_0(\tau),\tau)T_0(\tau)d\tau + \int_{-\infty}^{h_0(0)} G_2(x,t;\xi,0)T^0(\xi)d\xi, \tag{2.5}$$

with

$$v_1(t) = T_x(h_0(t)-,t).$$

We differentiate both sides of (2.5) with respect to x, then take $x \to h_0(t)-$. To go into the details, because $D_O G_{2\xi x} = -D_O G_{2\xi\xi} = G_{2\tau}$, we have

$$- D_O \int_0^t G_{2\xi x}(x,t;h_0(\tau),\tau)T_0(\tau)d\tau = -\int_0^t G_{2\tau}(x,t;h_0(\tau),\tau)T_0(\tau)d\tau$$
$$= -\int_0^t \left[\frac{\partial}{\partial\tau}\left(G_2(x,t;h_0(\tau),\tau)\right) - G_{2\xi}(x,t;h_0(\tau),\tau)h_0'(\tau)\right]T_0(\tau)d\tau$$
$$= G_2(x,t;h_0(0),0)T_0(0) + \int_0^t G_2(x,t;h_0(\tau),\tau)T_0'(\tau)d\tau$$
$$- \int_0^t G_{2x}(x,t;h_0(\tau),\tau)T_0(\tau)h_0'(\tau)d\tau.$$

Moreover,

$$\int_{-\infty}^{h_0(0)} G_{2x}(x,t;\xi,0)T^0(\xi)d\xi = -\int_{-\infty}^{h_0(0)} G_{2\xi}(x,t;\xi,0)T^0(\xi)d\xi$$
$$= -G_2(x,t;h_0(0),0)T^0(h_0(0)) + \int_{-\infty}^{h_0(0)} G_2(x,t;\xi,0)T^0_\xi(\xi)d\xi.$$

Using the compatible condition $T_0(0) = T^0(h_0(0))$, we have, for all $h_0(t) > x > -\infty$

$$T_x(x,t) = D_O \int_0^t G_{2x}(x,t;h_0(\tau),\tau)v_1(\tau)d\tau$$
$$+ \int_0^t G_2(x,t;h_0(\tau),\tau)T_0'(\tau)d\tau + \int_{-\infty}^{h_0(0)} G_2(x,t;\xi,0)T^0_\xi(\xi)d\xi. \tag{2.6}$$

Taking $x \to h_0(t)-$ and using Lemma 2.1 for the first term, we have

$$v_1(t) = 2D_O \int_0^t G_{2x}(h_0(t), t; h_0(\tau), \tau) v_1(\tau) d\tau + 2 \int_0^t G_2(h_0(t), t; h_0(\tau), \tau) T_0'(\tau) d\tau + 2 \int_{-\infty}^{h_0(0)} G_2(h_0(t), t; \xi, 0) T_\xi^0(\xi) d\xi. \tag{2.7}$$

Now we consider the heat distribution in false-bottoms. Apply Lemma 2.1 to equation (1.5), for $h_u(t) > x > h_0(t)$,

$$\begin{aligned} T(x,t) &= D_I \int_0^t G_3(x, t; h_u(\tau), \tau) v_3(\tau) d\tau \\ &- \int_0^t G_3(x, t; h_0(\tau), \tau) \left[D_I v_2(\tau) + T_0(\tau) h_0'(\tau) \right] d\tau \\ &+ D_I \int_0^t G_{3\xi}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau + \int_{h_0(0)}^{h_u(0)} G_3(x, t; \xi, 0) T^0(\xi) d\xi, \end{aligned} \tag{2.8}$$

where

$$v_2(t) = T_x(h_0(t)+, t), \;\; v_3(t) = T_x(h_u(t)-, t) = \frac{1}{\widetilde{\lambda}_I} h_u'(t).$$

Let us differentiate both sides of (2.8) with respect to x. We have the same calculation to (2.5):

$$\begin{aligned} D_I \int_0^t G_{3\xi x}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau &= \int_0^t G_{3\tau}(x, t; h_0(\tau), \tau) T_0(\tau) d\tau \\ = \int_0^t \left[\frac{\partial}{\partial \tau} \left(G_3(x, t; h_0(\tau), \tau) \right) - G_{3\xi}(x, t; h_0(\tau), \tau) h_0'(\tau) \right] T_0(\tau) d\tau \\ = -G_3(x, t; h_0(0), 0) T_0(0) - \int_0^t G_3(x, t; h_0(\tau), \tau) T_0'(\tau) d\tau \\ + \int_0^t G_{3x}(x, t; h_0(\tau), \tau) T_0(\tau) h_0'(\tau) d\tau, \end{aligned}$$

and

$$\int_{h_0(0)}^{h_u(0)} G_{3x}(x,t;\xi,0)T^0(\xi)d\xi = -\int_{h_0(0)}^{h_u(0)} G_{3\xi}(x,t;\xi,0)T^0(\xi)d\xi$$

$$= G_3(x,t;h_0(0),0)T^0(h_0(0)) + \int_{h_0(0)}^{h_u(0)} G_3(x,t;\xi,0)T^0_\xi(\xi)d\xi.$$

Here we have used the compatible condition $T^0(h_u(0)) = 0$. Using again the compatible condition $T^0(h_0(0)) = T_0(0)$, we find that

$$T_x(x,t) = D_I \int_0^t G_{3x}(x,t;h_u(\tau),\tau)v_3(\tau)d\tau - D_I \int_0^t G_{3x}(x,t;h_0(\tau),\tau)v_2(\tau)d\tau$$
$$- \int_0^t G_3(x,t;h_0(\tau),\tau)T_0'(\tau)d\tau + \int_{h_0(0)}^{h_u(0)} G_3(x,t;\xi,0)T^0_\xi(\xi)d\xi. \tag{2.9}$$

for all $h_u(t) > x > h_0(t)$. Taking $x \to h_0(t)+$ and $x \to h_u(t)-$ both sides of (2.9) and using (Lemma 2.2) one has

$$v_2(t) = 2D_I \int_0^t G_{3x}(h_0(t),t;h_u(\tau),\tau)v_3(\tau)d\tau - 2D_I \int_0^t G_{3x}(h_0(t),t;h_0(\tau),\tau)v_2(\tau)d\tau$$
$$-2\int_0^t G_3(h_0(t),t;h_0(\tau),\tau)T_0'(\tau)d\tau + 2\int_{h_0(0)}^{h_u(0)} G_3(h_0(t),t;\xi,0)T^0_\xi(\xi)d\xi, \tag{2.10}$$

and

$$v_3(t) = 2D_I \int_0^t G_{3x}(h_u(t),t;h_u(\tau),\tau)v_3(\tau)d\tau - 2D_I \int_0^t G_{3x}(h_u(t),t;h_0(\tau),\tau)v_2(\tau)d\tau$$
$$-2\int_0^t G_3(h_u(t),t;h_0(\tau),\tau)T_0'(\tau)d\tau + 2\int_{h_0(0)}^{h_u(0)} G_3(h_u(t),t;\xi,0)T^0_\xi(\xi)d\xi. \tag{2.11}$$

Four equations (2.4), (2.7), (2.10) and (2.11) form a system of nonlinear integral equations with respect to $S_0(t)$, $v_1(t)$, $v_2(t)$ and $v_3(t)$. Here, due to the interface condition (1.1) and (1.4), one has

$$h_0(t) = h_0(0) + \widetilde{\lambda}_I \int_0^t v_2(\tau)d\tau - \widetilde{\lambda}_O \int_0^t v_1(\tau)d\tau. \tag{2.12}$$

and

$$h_u(t) = h_u(0) + \widetilde{\lambda}_I \int_0^t v_3(\tau)d\tau. \tag{2.13}$$

We thus have proved the direct part of the following statement.

Theorem 2.2. *The problem (1.1)-(1.7) in* $[0, \sigma]$ *is equivalent to the problem of finding a continuous solution* $v = (S_0, v_1, v_2, v_2)$ *in* $[0, \sigma]$ *for the system (2.4), (2.7), (2.10) and (2.11), where* h_0 *and* h_u *are given by (2.12) and (2.13) and satisfy* $h_u(t) > h_0(t)$ *in* $[0, \sigma)$.

Of course, we only need to prove the converse part of the statement.

Proof. Suppose that $v = (S_0, v_1, v_2, v_2)$ is a continuous solution in $[0, \sigma]$ for the system (2.4), (2.7), (2.10) and (2.11), with h_0 and h_u are given by (2.12) and (2.13) and satisfy $h_u(t) > h_0(t)$ in $[0, \sigma]$. Define $S(x, t)$, $T(x, t)$ by (2.3), (2.5) and (2.8). We want to check (h_0, h_u, T, S) satisfies (C1)-(C4).

It is clear that (C1) is automatically satisfied. Because $S(x, t)$ and $T(x, t)$ are defined by integral forms (2.3), (2.5) and (2.8), the smoothness conditions in (C2)-(C3), excepting the continuity of $T(., t)$ at $x = h_0(t)$, hold. The initial conditions in (C4) simply follow by getting $t \to 0+$ in (2.12), (2.13), (2.3), (2.5) and (2.8) and using

$$\lim_{t \to 0+} G(x, t; \xi, 0) = \delta(x - \xi),$$

with $G = G_1, G_2, G_3$. It remains to check equations (1.1)-(1.7) and prove that $T(h_0(t), t) = T_0(t)$.

Step 1. We start by prove three diffusion equations (1.5)-(1.7). To prove (1.5), where $T(x, t)$ is defined by (2.8) in $h_u(t) > x > h_0(t)$, we shall verify that each of four terms in the right-hand side of (2.8) is a homogeneous solution of the operator $(\partial/\partial t - D_I \partial^2/\partial x^2)$. This fact is clear for the fourth term due to the property of Green function,

$$\left(\frac{\partial}{\partial t} - D_I \frac{\partial^2}{\partial x^2}\right) \left(\int_{h_0(0)}^{h_u(0)} G_3(x, t; \xi, 0) T^0(\xi) d\xi \right)$$

$$= \int_{h_0(0)}^{h_u(0)} \left(\frac{\partial}{\partial t} - D_I \frac{\partial^2}{\partial x^2}\right) (G_3(x, t; \xi, 0))\, T^0(\xi) d\xi = 0$$

For the three first terms, we have to be more careful because in general differentiating with respect to t a function of the form $t \mapsto \int_0^t K(t, \tau) d\tau$ may cause a jump,

$$\frac{d}{dt} \left(\int_0^t K(t, \tau) d\tau \right) = \lim_{\tau \to t-} K(t, \tau) + \int_0^t K_t(t, \tau) d\tau.$$

However, in this case the jump $\lim\limits_{\tau\to t-} K(t,\tau)$ vanishes due to

$$\lim_{\tau\to t-} G_3(x,t;\xi,\tau) = \lim_{\tau\to t-} G_{3\xi}(x,t;\xi,\tau) = 0, \quad \forall x \neq \xi.$$

Therefore, the three first terms can be treat similarly to the fourth term. Thus (1.5) holds. And (1.6)-(1.7) can be treated by the same way.

Step 2. Next, we check that S fits the Dirichlet condition $S(h_0(t)-,t) = S_0(t)$, T fits three Neumann conditions $T_x(h_0(t)-,t) = v_1(t)$, $T_x(h_0(t)+,t) = v_2(t)$ and $T_x(h_u(t)-,t) = v_3(t)$, and deduce the Stefan conditions (1.1) and (1.4).

Infact, from (2.3) taking $x \to h_0(t)-$ and using Lemma 2.2 one has

$$S(h_0(t)-,t) = \frac{1}{2}S_0(t) - D\int_0^t G_{1\xi}(h_0(t),t;h_0(\tau),\tau)S_0(\tau)d\tau$$
$$+ \int_{-\infty}^{\infty} G_1(h_0(t),t;\xi,0)S(\xi,0)d\xi.$$

Comparing the latter equation to(2.4), we conclude that $S(h_0(t)-,t) = S_0(t)$.

Differentiate both sides of (1.5) with respect to x and then take $x \to h_u(t)-$. Using the same process of geting (2.11) from (2.8), one has

$$T_x(h_u(t)-,t) = \frac{1}{2}v_3(t) + D_I\int_0^t G_{3x}(h_u(t),t;h_u(\tau),\tau)v_3(\tau)d\tau$$
$$-D_I\int_0^t G_{3x}(h_u(t),t;h_0(\tau),\tau)v_2(\tau)d\tau$$
$$-\int_0^t G_3(h_u(t),t;h_0(\tau),\tau)T_0'(\tau)d\tau + \int_{h_0(0)}^{h_u(0)} G_3(h_u(t),t;\xi,0)T_\xi^0(\xi)d\xi.$$

Comparing the latter equation to(2.11), we find that $T_x(h_u(t)-,t) = v_3(t)$. Similarly, $T_x(h_0(t)+,t) = v_2(t)$ and $T_x(h_0(t)-,t) = v_1(t)$. Thus the Stefan conditions (1.1) and (1.4) follow the definitions of $h_0(t)$ and $h_u(t)$ in (2.12) and (2.13).

Step 3. Finally, we show that S fits the Stefan condition (1.2), T fits the Dirichlet conditions (1.3) and $T(h_0(t),t) = T_0$.

Applying Lemma 2.1 to equation (1.7), for $h_0(t) > x > -\infty$ and using $S(h_0(t)-,t) = S_0(t)$, one has

$$S(x,t) = \int_0^t G_1(x,t;h_0(\tau),\tau)\left[DS_x(h_0(\tau)-,\tau) + S_0(\tau)h_0'(\tau)\right]d\tau$$
$$-D\int_0^t G_{1\xi}(x,t;h_0(\tau),\tau)S_0(\tau)d\tau + \int_{-\infty}^{\infty} G_1(x,t;\xi,0)S(\xi,0)d\xi.$$

Comparing to the original definition of S in (2.3), we deduce

$$\int_0^t G_1(x,t;h_0(\tau),\tau)\Psi_1(\tau)d\tau = 0,\;\; h_0(t) > x > -\infty,$$

where $\Psi_1(t) = DS_x(h_0(t)-,t) + S_0(t)h_0'(t)$. We next differentiate the latter equation with respect to x, then take $x \to h_0(t)-$ and use Lemma 2.2 to get

$$\Psi_1(t) = -2\int_0^t G_{1x}(h_0(t),t;h_0(\tau),\tau)\Psi_1(\tau)d\tau. \tag{2.14}$$

This is a linear Volterra integral equation of the second kind with respect to $\Psi_1(t)$, and

$$|G_{1x}(h_0(t),t;h_0(\tau),\tau)| \leq \frac{const.}{\sqrt{t-\tau}}.$$

It follows from Lemma 2.3 that $\Psi_1(t) = 0$. Thus (1.2) holds.

Similarly, applying Lemma 2.1 to equation (1.6), for $h_0(t) > x > -\infty$, one has

$$\begin{aligned} T(x,t) = & \int_0^t G_2(x,t;h_0(\tau),\tau)\left[D_O T_x(h_0(\tau)-,\tau) + T(h_0(\tau)-,\tau)h_0'(\tau)\right]d\tau \\ & - D_O\int_0^t G_{2\xi}(x,t;h_0(\tau),\tau)T(h_0(\tau)-,\tau)d\tau + \int_{-\infty}^{h_0(0)} G_2(x,t;\xi,0)T^0(\xi)d\xi, \end{aligned}$$

Comparing the latter equation to the original definition of T in (2.5), and using $T_x(h_0(t)-,t) = v_1(t)$, we obtain

$$\int_0^t \left[G_2(x,t;h_0(\tau),\tau)h_0'(\tau) - D_O G_{2\xi}(x,t;h_0(\tau),\tau)\right]\Psi_2(\tau)d\tau = 0, \tag{2.15}$$

where $\Psi_2(t) = T(h_0(t)-,t) - T_0(t)$. Taking $x \to h_u(t)-$ in (2.15) and using Lemma 2.2, we find that

$$\Psi_2(t) = 2\int_0^t \left[G_2(x,t;h_0(\tau),\tau)h_0'(\tau) - D_O G_{2\xi}(x,t;h_0(\tau),\tau)\right]\Psi_2(\tau)d\tau. \tag{2.16}$$

This is a linear Voterra integral equation of the second kind. Since

$$|G_2(x,t;h_0(\tau),\tau)h_0'(\tau) - D_O G_{2\xi}(x,t;h_0(\tau),\tau)| \leq \frac{const.}{\sqrt{t-\tau}},$$

it follows from Lemma 2.3 that (2.16) has a unique solution $\Psi_2(t) = 0$. Thus $T(h_0(t)-,t) = T_0(t)$.

Applying Lemma 2.1 to equation (1.5), for $h_u(t) > x > h_0(t)$ and comparing to the original definition of T in (2.8), one has

$$\begin{aligned}&\int_0^t \left[G_3(x,t;h_0(\tau),\tau)h_0'(\tau) - D_I G_{3\xi}(x,t;h_0(\tau),\tau)\right]\Psi_3(\tau)d\tau \\ &= \int_0^t \left[G_3(x,t;h_u(\tau),\tau)h_u'(\tau) - D_I G_{3\xi}(x,t;h_u(\tau),\tau)\right]\Psi_4(\tau)d\tau,\end{aligned} \tag{2.17}$$

where $\Psi_3(\tau) = T(h_0(t)+,t) - T_0(t)$ and $\Psi_4(t) = T(h_u(t),t)$. Taking $x \to h_0(t)+$ and $x \to h_u(t)-$ in (2.17), we have

$$\begin{aligned}\Psi_3(t) =& \int_0^t \left[G_3(h_0(t),t;h_0(\tau),\tau)h_0'(\tau) - D_I G_{3\xi}(h_0(t),t;h_0(\tau),\tau)\right]\Psi_3(\tau)d\tau \\ &- \int_0^t \left[G_3(h_0(t),t;h_u(\tau),\tau)h_u'(\tau) - D_I G_{3\xi}(h_0(t),t;h_u(\tau),\tau)\right]\Psi_4(\tau)d\tau,\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\Psi_4(t) =& -\int_0^t \left[G_3(h_u(t),t;h_0(\tau),\tau)h_0'(\tau) - D_I G_{3\xi}(h_u(t),t;h_0(\tau),\tau)\right]\Psi_3(\tau)d\tau \\ &+ \int_0^t \left[G_3(h_u(t),t;h_u(\tau),\tau)h_u'(\tau) - D_I G_{3\xi}(h_u(t),t;h_u(\tau),\tau)\right]\Psi_4(\tau)d\tau.\end{aligned} \tag{2.19}$$

Equations (2.18) and (2.19) form a system of linear Volterra integral equations of the second kind, and we deduce from Lemma 2.3 that $\Psi_3 = \Psi_4 = 0$. Thus (1.3) holds, and the continuity of $T(.,t)$ at $h_0(t)$ follows $T(h_0(t)+,t) = T(h_0(t)-,t) = T_0(t)$. □

Remark 2.4. *The technique in Step 3 is learnt from A.Friedman [3]. However, there is a minor mistake in [3]: formula (1.28) in page 221, i.e.*

$$\int_0^t G_\xi(x,t;s(\tau),\tau)u(s(\tau),\tau)d\tau = 0,$$

should be

$$\int_0^t \left[G_\xi(x,t;s(\tau),\tau) - G(x,t;s(\tau),\tau)s'(\tau)\right]u(s(\tau),\tau)d\tau = 0.$$

Fortunately, the conclusion $u(s(t),t) = 0$ still holds by the same argument.

2.3 Solving integral equations

Now we accomplish the proof of Theorem 2.1. We want to prove the system (2.3), (2.7), (2.10), (2.11) has a unique local solution, and this solution can be extended uniquely by prolongation whenever the condition $h_u(t) > h_0(t)$ still holds. The system (2.3), (2.7), (2.10), (2.11) can be rewritten as $v = Pv$, where

$$v = (S_0, v_1, v_2, v_3)\,, Pv = (P_1v, P_2v, P_3v, P_4v)\,.$$

Denote by $C(\sigma, M)$ the space of functions $v = (S_0, v_1, v_2, v_3)$ continuous in $[0, \sigma]$ and

$$\|v\|_{[0,\sigma]} := \sup_{t\in[0,\sigma]} \max\{|S_0(t)|\,, |v_1(t)|\,, |v_2(t)|\,, |v_3(t)|\} \le M.$$

We shall prove that for $M > 0$ large enough, says $M \ge 2\,\|P(0)\|_{[0,\sigma]}$, there exists $t_M > 0$ depending on M (but independent on $v \in C(t_M, M)$) such that P is a contraction on $C(t_M, M)$.

First at all, note that for each $M > 0$, if $t_M > 0$ small enough then it follows from (2.12)-(2.13) that

$$h_u(t_1) - h_0(t_2) \ge \frac{h_u(0) - h_0(0)}{2} > 0,\ \forall t_1, t_2 \in [0, t_M]. \tag{2.20}$$

In particular, $h_u(t) > h_0(t)$ for all $t \in [0, t_M]$, and hence P is well-defined.

Let us estimate $\|Pv - P\widetilde{v}\|_{[0,t_M]}$ for $v, \widetilde{v} \in C(t_M, M)$. In what follows, denote by $\widetilde{h}_u$ and $\widetilde{h}_0$ the functions given by (2.12)-(2.13) where v is replaced by $\widetilde{v}$; and denote by $C_0 > 0$ a constant depending only on data $(h_0(0), h_u(0), T^0(x), S^0(x), T_0(t))$ but independent on M, v and $\widetilde{v}$. We shall go to the details for P_4, and P_1, P_2, P_3 can be treated by the same way.

For preparation, we need some estimates related to Green's function.

Lemma 2.4. *Let $v, \widetilde{v} \in C(t_M, M)$. Then for $t > 0$ small enough we have*

$$\begin{aligned}
&\left|G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_u(\tau), \tau)\right| \le \frac{C_0 M}{\sqrt{t-\tau}}\,\|v - \widetilde{v}\|_{[0,t_M]}\,,\\
&\left|G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\widetilde{h}_u(t), t; \widetilde{h}_0(\tau), \tau)\right| \le t\,\|v - \widetilde{v}\|_{[0,t_M]}\,,\\
&\left|G_{3x}(h_u(t), t; h_0(\tau), \tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_0(\tau), \tau)\right| \le t\,\|v - \widetilde{v}\|_{[0,t_M]}\,,\\
&\left|G_{3x}(h_u(t), t; \xi, \tau) - G_{3x}(\widetilde{h}_u(t), t; \xi, \tau)\right| \le\\
&\le C_0\left[\exp\left(-\frac{(h_u(t)-\xi)^2}{4D_I t}\right) + \exp\left(-\frac{(h_u(t)-\xi)^2}{4D_I t}\right)\right]\|v - \widetilde{v}\|_{[0,t_M]}\,.
\end{aligned}$$

for all $t \in [0, t_M]$ and $\xi \in \mathbb{R}$.

Proof. First at all, we note that h_0 ,h_u are linear with respect to v and

$$|h_u(t) - h_u(\tau)| = \left|\widetilde{\lambda}_I \int_\tau^t v_3(r)dt\right| \le \widetilde{\lambda}_I\,\|v_3\|_{[0,t_M]}\,(t-\tau),$$

$$|h_0(t) - h_0(\tau)| \le \left[\widetilde{\lambda}_O\,\|v_1\|_{[0,t_M]} + \widetilde{\lambda}_O\,\|v_2\|_{[0,t_M]}\right](t-\tau).$$

Let us now consider function

$$G_{3x}(h_u(t),t;h_u(\tau),\tau)=\frac{h_u(t)-h_u(\tau)}{4\sqrt{\pi D_I^3(t-\tau)^3}}\exp\left(-\frac{(h_u(t)-h_u(\tau))^2}{4D_I(t-\tau)}\right).$$

as a one-variable function of $(h_u(t)-h_u(\tau))$. Using Lagrange formula one has

$$\begin{aligned}&G_{3x}(h_u(t),t;h_u(\tau),\tau)-G_{3x}(\widetilde{h}_u(t),t;\widetilde{h}_u(\tau),\tau)\\ =&\frac{\left[(h_u(t)-h_u(\tau))-\left(\widetilde{h}_u(t)-\widetilde{h}_u(\tau)\right)\right]}{4\sqrt{\pi D_I^3(t-\tau)^3}}\exp\left(-\frac{\theta^2}{4D_I(t-\tau)}\right)\left(1-\frac{2\theta^2}{4D_I(t-\tau)}\right).\end{aligned}$$

The first inequality follows $|\theta|\leq\widetilde{\lambda}_I M(t-\tau)$ and

$$\left|(h_u(t)-h_u(\tau))-\left(\widetilde{h}_u(t)-\widetilde{h}_u(\tau)\right)\right|\leq\widetilde{\lambda}_I(t-\tau)\left\|v-\widetilde{v}\right\|_{[0,t_M]}.$$

By the same argument, one has

$$\begin{aligned}&G_3(h_u(t),t;h_0(\tau),\tau)-G_3(\widetilde{h}_u(t),t;h_0(\tau),\tau)\\ =&\frac{\left[h_u(t)-\widetilde{h}_u(t)\right]\theta}{4\sqrt{\pi D_I^3(t-\tau)^3}}\exp\left(-\frac{\theta^2}{4D_I(t-\tau)}\right),\end{aligned}$$

and

$$\begin{aligned}&G_{3x}(h_u(t),t;h_0(\tau),\tau)-G_{3x}(\widetilde{h}_u(t),t;\widetilde{h}_0(\tau),\tau)\\ =&\frac{h_u(t)-\widetilde{h}_u(t)}{4\sqrt{\pi D_I^3(t-\tau)^3}}\exp\left(-\frac{\theta^2}{4D_I(t-\tau)}\right)\left(1-\frac{2\theta^2}{4D_I(t-\tau)}\right)\end{aligned}$$

for some θ between $(h_u(t)-h_0(\tau))$ and $(\widetilde{h}_u(t)-\widetilde{h}_0(\tau))$. Note that

$$|h_u(t)-h_{1u}(t)|\leq\widetilde{\lambda}_I M\left\|v-v_1\right\|_{[0,t]},$$

and it follows from (2.20) that

$$|\theta|\geq\frac{h_u(0)-h_0(0)}{2}>0.$$

Therefore, the second and the third inequalities follow the elementary inequalities $z^3e^{-z}\leq const.$, and $(z^3+z^4)e^{-z}\leq const.$, for $z>0$.

For the last inequality of Lemma 2.4, we write

$$\begin{aligned}&G_3(h_u(t),t;\xi,0)-G_3(h_{1u}(t),t;\xi,0)\\ =&\frac{1}{2\sqrt{\pi D_I t}}\left[\exp\left(-\frac{(\xi-h_u(t))^2}{4D_I(t-\tau)}\right)-\exp\left(-\frac{(\xi-h_u(t))^2}{4D_I(t-\tau)}\right)\right].\end{aligned}$$

and then employ the following elementary inequality

$$\left|\exp(-z_1^2)-\exp(-z_2^2)\right|\leq const.\left[\exp\left(-\frac{z_1^2}{2}\right)+\exp\left(-\frac{z_1^2}{2}\right)\right]|z_1-z_2| \qquad (2.21)$$

for $z_1, z_2 \in \mathbb{R}$. To prove (2.21), we can assume $0 \le z_1 < z_2$ and note that

$$\frac{\exp(-z_1^2) - \exp(-z_2^2)}{z_1 - z_2} = -2\theta \exp(-\theta^2)$$

for some $\theta \in (z_1, z_2$. Then (2.21) follows $|\theta| \exp(-\theta^2/2) \le const.$, and $\exp(-\theta^2/2) \le \exp(-z_2^2/2)$. □

Return to estimate $||P_4 v - P_4 \widetilde{v}||_{[0,t_M]}$. Using the first inequality of Lemma 2.4, we get

$$\begin{aligned}
&\left| G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_u(\tau), \tau) \widetilde{v}_3(\tau) \right| \\
&\le \left| G_{3x}(h_u(t), t; h_u(\tau), \tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_u(\tau), \tau) \right| |v_3(\tau)| \\
&\quad + |G_{3x}(h_u(t), t; h_u(\tau), \tau)| \, |v_3(\tau) - \widetilde{v}_3(\tau)| \\
&\le \frac{C_0 M}{\sqrt{t-\tau}} \|v - \widetilde{v}\|_{[0,t_M]} \times M + \frac{C_0 M}{\sqrt{t-\tau}} \times \|v - \widetilde{v}\|_{[0,t_M]} .
\end{aligned}$$

Therefore

$$\begin{aligned}
&2D_I \int_0^t \left| G_{3x}(h_u(t), t; h_u(\tau), \tau) v_3(\tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_u(\tau), \tau) \widetilde{v}_3(\tau) \right| d\tau \\
&\le 2D_I \int_0^t \frac{C_0(M^2 + M)}{\sqrt{t-\tau}} \|v - \widetilde{v}\|_{[0,t_M]} \, d\tau = 4D_I C_0 (M^2 + M) \sqrt{t_M} \, \|v - \widetilde{v}\|_{[0,t_M]} .
\end{aligned} \tag{2.22}$$

Also, from the second and the third inequality of Lemma 2.4, we have

$$\begin{aligned}
&2D_I \int_0^t \left| G_{3x}(h_u(t), t; h_0(\tau), \tau) v_2(\tau) - G_{3x}(\widetilde{h}_u(t), t; \widetilde{h}_0(\tau), \tau) \widetilde{v}_2(\tau) \right| d\tau \\
&\le 2D_I \int_0^t t(M+1) \|v - \widetilde{v}\|_{[0,t_M]} d\tau = 2D_I (M+1) t_M^2 \, \|v - \widetilde{v}\|_{[0,t_M]}
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&2D_I \int_0^t \left| G_3(h_u(t), t; h_0(\tau), \tau) - G_3(\widetilde{h}_u(t), t; \widetilde{h}_0(\tau), \tau) \right| T_0'(\tau) d\tau \\
&\le 2D_I \int_0^t t \, \|v - \widetilde{v}\|_{[0,t_M]} \, T_0'(\tau) d\tau \le C_0 t \, \|v - \widetilde{v}\|_{[0,t_M]} .
\end{aligned} \tag{2.24}$$

It follows from the last inequality of Lemma 2.4 that

$$2\int\limits_{h_0(0)}^{h_u(0)} \left| G_3(h_u(t),t;\xi,0) - G_3(\widetilde{h}_u(t),t;\xi,0)\right| T^0_\xi(\xi)d\xi$$

$$\leq C_0 \left\| v - \widetilde{v}\right\|_{[0,t_M]} \int\limits_{h_0(0)}^{h_u(0)} \left[\exp\left(-\frac{(\xi - h_u(t))^2}{C_0 t}\right) + \exp\left(-\frac{(\xi - \widetilde{h}_u(t))^2}{C_0 t}\right)\right] d\xi.$$

On the other hand, by making some change of variables, one has

$$\int\limits_{-\infty}^{\infty} \left[\exp\left(-\frac{(\xi - h_u(t))^2}{C_0 t}\right) + \exp\left(-\frac{(\xi - \widetilde{h}_u(t))^2}{C_0 t}\right)\right] d\xi$$

$$= 2\int\limits_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{C_0 t}\right) d\xi = 2\sqrt{\pi C_0 t} \left\| v - \widetilde{v}\right\|_{[0,t_M]}.$$

Thus

$$\begin{aligned} &2\int\limits_{h_0(0)}^{h_u(0)} \left| G_3(h_u(t),t;\xi,0) - G_3(\widetilde{h}_u(t),t;\xi,0)\right| T^0_\xi(\xi)d\xi \\ &\leq 2\sqrt{\pi C_0^3 t_M} \left\| v - \widetilde{v}\right\|_{[0,t_M]}. \end{aligned} \tag{2.25}$$

It follows from (2.22)-(2.25) that

$$\left\| P_4 v - P_4\widetilde{v}\right\|_{[0,t_M]} \leq C_0 M^2 \sqrt{t_M} \left\| v - \widetilde{v}\right\|_{[0,t_M]},$$

where C_0 stands for a constant depending only on data $(h_0(0), h_u(0), T^0(x), S^0(x), T_0(t))$. We also have similar estimates for P_1, P_2, P_3. Thus, if

$$M^2\sqrt{t_M} \leq \varepsilon_0, \tag{2.26}$$

where ε_0 stands for a constant depending only on data $(h_0(0), h_u(0), T^0(x), S^0(x), T_0(t))$, then

$$\left\| Pv - P\widetilde{v}\right\|_{[0,t_M]} \leq \frac{1}{2}\left\| v - \widetilde{v}\right\|_{[0,t_M]}, \ \forall v, \widetilde{v} \in C(t_M, M). \tag{2.27}$$

The latter inequality implies that if $M \geq 2\left\| P(0)\right\|_{[0,\sigma]}$ then P maps from $C(t_M, M)$ into itself and is a contraction. Thus P has a unique fixed point v in $C(t_M, M)$ for t_M small enough (restricted only by (2.26)).

Now let us prove the solution can be extended uniquely by prolongation. Assume that v is a solutions of system (2.3), (2.7), (2.10), (2.11) in $[0, \sigma^*]$ with $\sigma^* < \sigma$ and $h_u(\sigma^*) > h_0(\sigma^*)$. Let us consider the system (2.3), (2.7), (2.10), (2.11) for $t \geq \sigma^*$ (instead of $t \geq 0$ under the data

$$(h_0(\sigma^*), h_u(\sigma^*), T(x, \sigma^*), S(x, \sigma^*), T_0(t)).$$

We need to check that the new data satisfy Hypothesis (H1)-(H4). Infact, (H1) and (H4) are automatically satisfied because $h_u(\sigma^*) > h_0(\sigma^*)$ and $T_0(t)$ is even continuously differentiable in $[0, \sigma]$. For (H2), the continuity of $T(., \sigma^*)$ and $T_x(., \sigma^*)$ is guaranteed by the definition of the solution. The boundedness of $T_x(., \sigma^*)$ follows the finite limits

$$\begin{aligned} T_x(h_0(\sigma^*)-, \sigma^*) &= v_1(\sigma^*), \\ T_x(h_0(\sigma^*)+, \sigma^*) &= v_2(\sigma^*), \\ T_x(h_u(\sigma^*)-, \sigma^*) &= v_3(\sigma^*), \end{aligned}$$

and, from (2.6),

$$\limsup_{x\to-\infty} |T_x(x, \sigma^*)| = \limsup_{x\to-\infty} \left| \int_{-\infty}^{h_0(0)} G_2(x, t; \xi, 0) T_\xi^0(\xi) d\xi \right| \leq \sup_{h_0(0)>\xi>-\infty} \left| T_\xi^0(\xi) \right|.$$

Here we have used a property of Green's function

$$\int_{-\infty}^{\infty} |G_2(x, t; \xi, 0)| \, d\xi = 1.$$

Similarly, (H3) holds. Thus Hypothesis (H1)-(H4) holds for the new data

$$(h_0(\sigma^*), h_u(\sigma^*), T(x, \sigma^*), S(x, \sigma^*), T_0(t)).$$

Therefore, it follows from the above result that the solution can be extended uniquely in $[0, \sigma^{**}]$ for some $\sigma^{**} \in (\sigma^*, \sigma]$.

Remark 2.5. *It is sometimes difficult to measure the initial condition* $(T(x, 0), S(x, 0))$ *in the infinite domain* $h_u(0) > x > -\infty$ *as in the previous section. However, this condition is maybe replaced by the initial condition* $(T(x, 0), S(x, 0))$ *in a finite interval* $h_u(0) > x > L$ *and the history information at one point* $(T(L, t), S(L, t))$ *in* $0 < t < \sigma$*, where* L *is a deep point in the ocean satisfying* $L < h_0(t)$ *for* $0 \leq t \leq \sigma$*.*

Infact, introduce the Green's function for the half-plane $x > L$*,*

$$K_j(x, t; \xi, \tau) = G_j(x, t; \xi, \tau) - G_j(2L - x, t; \xi, \tau), \ j = 1, 2.$$

Note that $K_j(x, t; L, \tau) = 0$ *and the formulae in Lemma 2.1 and Lemma 2.2 still hold with* K_j *replacing* G_j *and* $a(t) \equiv L$*. Thus instead of (2.3) and (2.5) we have, for* $h_0(t) > x > L$*,*

$$\begin{aligned} S(x, t) = &-D \int_0^t K_{1\xi}(x, t; h_0(\tau), \tau) S_0(\tau) d\tau + D \int_0^t K_{1\xi}(x, t; L, \tau) S(L, \tau) d\tau \\ &+ \int_L^{h_0(0)} K_1(x, t; \xi, 0) S(\xi, 0) d\xi, \end{aligned} \tag{2.28}$$

and

$$T(x,t) = \int_0^t K_2(x,t;h_0(\tau),\tau)\,[D_O T_\xi(h_0(\tau)-,\tau) + T_0(\tau)h_0'(\tau)]\,d\tau$$
$$- D_O \int_0^t K_{2\xi}(x,t;h_0(\tau),\tau)T_0(\tau)d\tau + D_O \int_0^t K_{2\xi}(x,t;L,\tau)T(L,\tau)d\tau \quad (2.29)$$
$$+ \int_L^{h_0(0)} K_2(x,t;\xi,0)T(\xi,0)d\xi.$$

Therefore, the uniqueness of the solution (h_0, h_u, T, S) *of the system (1.1)-(1.7) can be proved by the same argument.*

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